Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 1 - Graphing on a Number Line

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Graph the following inequality on the number line provided.

$$15 < -5x < 65$$

2. Graph the following inequality on the number line provided.

$$8(2.5x + \frac{5}{4}) < 3x - \frac{x}{2}$$

3. Graph the following inequality on the number line provided.

$$(x - 7)(x + 12) > 20$$

ANSWERS

(1 pt) 1.

(2 pts) 2.

(3 pts) 3.

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 2 - Operations on Polynomials

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Simplify and write the answer in standard polynomial form

$$(x^2 + 3)(x - 1) + (3x^2 - x + 7)$$

2. Solve for all values of x.

$$(x - 3)(x + 4) = 2x^2 - 32$$

3. Simplify and write the answer in standard polynomial form.

$$\frac{(x - 1)^4 - 16}{x^2 - 2x + 5}$$

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 3 - Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If there are 3 red socks, 4 blue socks, and 5 green socks in a drawer. What is the minimum number of socks you need to take out to guarantee that you will get a pair of socks with matching colors?

2. How many ways can Adam, Betty, Charles, and David be arranged in a line if Charles does not want to be first in line and Betty does not want to be last in line?

3. Coach Louis wants to divide his 8 players into three teams, Team A, Team B, and Team C. Out of the three teams, he wants two teams to have 3 players, and the remaining team to have 2 players. How many distinct ways can he assign players into the three teams?

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 4 - Perimeter, Area, and Volume

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If the volume of a cube is 729, what is the perimeter of one of its faces?

2. A rectangular prism with integer side lengths has a volume of 168 and a height of 6. What is the perimeter of the base of the rectangular prism given that all of its side lengths are greater than 2.

3. Given a square with side length 7 in the plane, what is the area of the region defined by the set of points within a distance of 2 from the boundary of the square? Consider the space both inside and outside of the square. Give your answer in the form $a + b\pi$.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x.

$$7(3x+2)-2(7x+3)=3(2x+7)$$

2. Evaluate the following expression. Write your answer as a fraction.

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{2}{5}+\frac{1}{2}}$$

3. Andy, Bill, and Cindy want to see how many books they can read in a week. On the first day, Andy reads 1 book, and each day after, he reads two more books than the previous day. Bill also reads 1 book on the first day, but for each day after, he reads twice as many books than he does the previous day. Cindy reads the same number of books each day. If by the end of the 7 days, Cindy has read one less book than Andy and Bill combined, how many books does she read each day?

4. Find the smallest positive integer divisible by 7 such that the sum of its digits is 18.

5. How many pairs of positive integers (x, y) satisfy

$$x^2-y^2=1024$$

6. Natalie wants to color each face of a cube with one of 4 possible colors. How many ways can she color each face such that no two adjacent faces are the same color? Orientation matters - two colorings are considered distinct even if one can be rotated into the other.

7. How many integers satisfy the inequality

$$x^2-3x-36<18$$

8. The radius of a cylinder is half of its height. If the volume of the cylinder is $\frac{2\pi}{27}$, what is the largest possible distance between two points that lie on the cylinder.

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Team Round Answer Sheet

ANSWERS

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Answer Key

Round 1 - Evaluation and Order of Operations

1. $-13 < x < -3$
2. $x < -\frac{4}{7}$
3. $x < -13$ or $x > 8$

Round 2 - Operations on Polynomials

1. $x^3 + 2x^2 + 2x + 4$
2. $-4, 5$
3. $x^2 - 2x - 3$

Round 3 - Fractions...

1. 4
2. 14
3. 1680

Round 4 - Perimeter, Area, and Volume

1. 36
2. 22
3. $96 + 4\pi$

Team Round

1. 13
2. $\frac{25}{54}$
3. 25
4. 189
5. 4
6. 96
7. 14
8. $\frac{2\sqrt{2}}{3}$

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 1 - Graphing on a Number Line

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Graph the following inequality on the number line provided.

$$15 < -5x < 65$$

Solution: Dividing by -5 flips the inequalities, so we get

$$\boxed{-13 < x < -3}$$

2. Graph the following inequality on the number line provided.

$$8(2.5x + \frac{5}{4}) < 3x - \frac{x}{2}$$

Solution: We have

$$\Rightarrow 20x + 10 < \frac{5}{2}x$$

$$\Rightarrow \frac{35}{2}x < -10$$

$$\Rightarrow \boxed{x < -\frac{4}{7}}$$

3. Graph the following inequality on the number line provided.

$$(x - 7)(x + 12) > 20$$

Solution: We have

$$\Rightarrow x^2 + 5x - 84 > 20$$

$$\Rightarrow x^2 + 5x - 104 > 0$$

$$\Rightarrow (x + 13)(x - 8) > 0$$

$$\Rightarrow \boxed{x < -13 \text{ or } x > 8}$$

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 2 - Operations on Polynomials

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Simplify and write the answer in standard polynomial form

$$(x^2+3)(x-1)+(3x^2-x+7)$$

Solution: We have

$$= (x^3-x^2+3x-3)+(3x^2-x+7)$$
$$= \boxed{x^3+2x^2+2x+4}$$

2. Solve for all values of x.

$$(x-3)(x+4) = 2x^2-32$$

Solution: We have

$$\Rightarrow x^2+x-12 = 2x^2-32$$
$$\Rightarrow x^2-x-20 = 0$$
$$\Rightarrow (x+4)(x-5)$$
$$\Rightarrow x = \boxed{-4, 5}$$

3. Simplify and write the answer in standard polynomial form.

$$\frac{(x-1)^4-16}{x^2-2x+5}$$

Solution: One solution would be to expand the numerator and divide. However, one could also recognize the difference of squares. We have

$$\begin{aligned}
&(x-1)^4 - 16 \\
&= ((x-1)^2)^2 - 4^2 \\
&= ((x-1)^2 - 4)((x-1)^2 + 4) \\
&= (x^2 - 2x - 3)(x^2 - 2x + 5)
\end{aligned}$$

Thus, the answer is $\boxed{x^2 - 2x - 3}$.

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 3 - Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If there are 3 red socks, 4 blue socks, and 5 green socks in a drawer. What is the minimum number of socks you need to take out to guarantee that you will get a pair of socks with matching colors? Thus, the minimum number of socks needed is $\boxed{4}$.

Solution: If she only takes out 3 socks, she could have taken one of each color. If she takes out 4 socks, she must have taken two of the same color by the Pigeonhole Principle.

2. How many ways can Adam, Betty, Charles, and David be arranged in a line if Charles does not want to be first in line and Betty does not want to be last in line?

Solution: This can be calculated by listing out all permutations and counting how many of them are satisfactory. We can also use the Principle of Inclusion-Exclusion. The number of ways is total permutations minus the number of ways Charles is first minus the number of ways Betty is last plus the number of ways Charles is first and Betty is last. This is $24 - 6 - 6 + 2 = \boxed{14}$.

3. Coach Louis wants to divide his 8 players into three teams, Team A, Team B, and Team C. Out of the three teams, he wants two teams to have 3 players, and the remaining team to have 2 players. How many distinct ways can he assign players into the three teams?

Solution: There are 3 ways to pick the team with 2 players. There are $\binom{8}{2}$ ways to choose 2 players for that team. There are $\binom{6}{3}$ ways to put the remaining players into two teams of 3. Thus, the total number of ways is

$$3\binom{8}{2}\binom{6}{3} = 3 \cdot 28 \cdot 20 = \boxed{1680}$$

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Round 4 - Perimeter, Area, and Volume

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If the volume of a cube is 729, what is the perimeter of one of its faces?

Solution: The side length of the cube is the cube root of 729, which is 9. The perimeter of one of its sides is $4 \cdot 9 = \boxed{36}$.

2. A rectangular prism with integer side lengths has a volume of 168 and a height of 6. What is the perimeter of the base of the rectangular prism given that all of its side lengths are greater than 2.

Solution: The area of the base is $168/6 = 28$. Since the sides are integers, the sides must be 1 and 28, 2 and 14, or 4 and 7. Given the condition that the side lengths are greater than 2, the sides must be 4 and 7, giving perimeter $2(4 + 7) = \boxed{22}$.

3. Given a square with side length 7 in the plane, what is the area of the region defined by the set of points within a distance of 2 from the boundary of the square? Consider the space both inside and outside of the square. Give your answer in the form $a + b\pi$.

Solution: This region is the interior of the square with a 3 by 3 square cut out plus 2 by 7 rectangles outside each side of the square plus quarter circles at each corner. This gives a total area of

$$7^2 - 3^2 + 4 \cdot 2 \cdot 7 + 4\frac{1}{4}\pi 2^2 = \boxed{96 + 4\pi}$$

Worcester County Mathematics League
Junior Varsity Meet 3 - March 27, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x.

$$7(3x+2) - 2(7x+3) = 3(2x+7)$$

Solution: We get

$$\Rightarrow 21x + 14 - 14x - 6 = 6x + 21$$
$$\Rightarrow x = \boxed{13}$$

2. Evaluate the following expression. Write your answer as a fraction.

$$\frac{\frac{3}{4} - \frac{1}{3}}{\frac{2}{5} + \frac{1}{2}}$$

Solution: We get

$$= \frac{\frac{9-4}{12}}{\frac{4+5}{10}}$$
$$= \frac{\frac{5}{12}}{\frac{9}{10}}$$
$$= \frac{5 \cdot 10}{9 \cdot 12}$$
$$= \boxed{\frac{25}{54}}$$

3. Andy, Bill, and Cindy want to see how many books they can read in a week. On the first day, Andy reads 1 book, and each day after, he reads two more books than the previous day. Bill also reads 1 book on the first day, but for each day after, he reads twice as many books than he does the previous day. Cindy reads the same number of books each day. If by the end of the 7 days, Cindy has read one less book than Andy and Bill combined, how many books does she read each day?

Solution: Andy reads $1+3+\ldots 13 = 7^2 = 49$ books. Bill reads $1+2+4+\cdots+63 = 2^7-1 = 127$ books. Thus, Cindy reads $49+127-1 = 175$ books. Then, Cindy read $\frac{175}{7} = \boxed{25}$ books a day.

4. Find the smallest positive integer divisible by 7 such that the sum of its digits is 18.

Solution: The smallest numbers that sum to 18 are $99, 189, 279, 288, \ldots$. We see that 189 is divisible by 7, so our answer is $\boxed{189}$.

5. How many pairs of positive integers (x, y) satisfy

$$x^2 - y^2 = 1024$$

Solution: This factors into $(x-y)(x+y) = 2^{10}$. Thus, we have $x-y = 2^i, x+y = 2^{10-i}$. We see that $i = 1, 2, 3, 4$ give us valid solutions, so there are $\boxed{4}$ solutions.

6. Natalie wants to color each face of a cube with one of 4 possible colors. How many ways can she color each face such that no two adjacent faces are the same color? Orientation matters - two colorings are considered distinct even if one can be rotated into the other.

Solution: There are 4 colors for the top. If the bottom side is one of the 3 other colors, the remaining sides must alternate in color, giving us 2 colorings. If the bottom side is the same color, there are 3 ways to color the front side. If the back side is the same color, there are $2 \cdot 2$ ways to color the remaining sides. If the back side is one of the 2 remaining colors, there is only 1 way to color the remaining sides. Thus, the number of colorings is

$$4(3 \cdot 2 + 3(2 \cdot 2 + 2)) = \boxed{96}$$

7. How many integers satisfy the inequality

$$x^2 - 3x - 36 < 18$$

Solution: We get

$$\Rightarrow x^2 - 3x - 54 < 0$$
$$\Rightarrow (x+6)(x-9) < 0$$
$$\Rightarrow -6 < x < 9$$

giving $\boxed{14}$ integer solutions.

8. The radius of a cylinder is half of its height. If the volume of the cylinder is $\frac{2\pi}{27}$, what is the largest possible distance between two points that lie on the cylinder.

Solution: We have $h = 2r$. We have $\frac{2\pi}{27} = \pi r^2 h = 2\pi r^3$, so $r = \frac{1}{3}, h = \frac{2}{3}$. The longest distance between two points on the cylinder is the hypotenuse of a right triangle with sides $2r$ and h, which is an isosceles right triangle with legs of length $\frac{2}{3}$. Thus, the distance is $\boxed{\frac{2\sqrt{2}}{3}}$.

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Team Round Answer Sheet

ANSWERS

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